CV, page #3, #16

On rings with near idempotent elements

[1]A. Badawi, [2]A. Y. M. Chin and [3]H. V. Chen

[1]*Department of Mathematics, Birzeit University,*
P.O. Box 14, Birzeit, West Bank, Palestine via Israel
[2,3]*Institute of Mathematical Sciences, Faculty of Science, University of Malaya,*
50603 Kuala Lumpur, Malaysia

E-mail: [1]*abring@birzeit.edu,* [2]*acym@mnt.math.um.edu.my*

Abstract. Let R be an associative ring with unit. An element $e \in R$ is said to be a *near idempotent* if e^n is an idempotent for some positive integer n. In this paper conditions on R which are equivalent to the condition that R has near idempotents as all its elements are obtained.

2001 Mathematics Subject Classification: primary 16E50; secondary 16U99

1 Introduction

All rings considered in this paper are associative with unit. Given a ring R, an element $e \in R$ is said to be a *near idempotent* if e^n is an idempotent for some positive integer n. Clearly, every idempotent is a near idempotent. We say that R is *Euler* if every element of R is a near idempotent. If there exists a fixed positive integer n such that x^n is an idempotent for every $x \in R$, then R is said to be *exact-Euler*. It is clear that an exact-Euler ring is Euler.

An element $x \in R$ is said to be *strongly π-regular* if there exist $y \in R$ and a positive integer n such that $x^n = x^{n+1}y$ and $xy = yx$ (see [1]). In the case where $n = 1$, x is said to be strongly regular. R is said to be a *strongly π-regular* ring if all its elements are strongly π-regular.

For a ring R we shall use $Id(R)$ and $U(R)$ to denote the set of idempotents and the set of units of R, respectively. The set of all nilpotent elements of R shall be denoted by $\mathrm{Nil}(R)$. In this paper we show that R is Euler iff R is strongly

π-regular and $U(R)$ is a torsion group. We also show that R is exact-Euler iff R is strongly π-regular and $\mathrm{Nil}(R)$, $U(R)$ are of bounded index. As a matter of interest we also give some results related to $(s,2)$-rings.

2 Some Preliminaries

Theorem 2.1 *Let R be a strongly π-regular ring. Then for each $x \in R$, there exists a positive integer n such that $x^n = eu = ue$ for some $e \in Id(R)$ and some $u \in U(R)$.*

Proof. Let $x \in R$. Since R is strongly π-regular, it follows that there exists a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. Then

$$x^n = x^{n+1}y = x^{n+2}y^2 = \cdots = x^{2n}y^n = x^n y^n x^n.$$

Let $e = x^n y^n$. Then $e^2 = (x^n y^n x^n)y^n = x^n y^n = e$ and e commutes with x and y. Note that

$$xye = xy(x^n y^n) = (x^{n+1}y)y^n = x^n y^n = e \tag{1}$$

and

$$x^n e = x^n(x^n y^n) = x^n. \tag{2}$$

Let $f = e + x(1-e)$. Since

$$\begin{aligned} f^n &= [e + x(1-e)]^n = e^n + x^n(1-e)^n \\ &= e + x^n(1-e) = e \qquad \text{(by (2))}, \end{aligned}$$

then f is a near idempotent. Let $v = xe + (1-e)$ and $w = ye + (1-e)$. Then

$$\begin{aligned} wv &= vw = [xe + (1-e)][ye + (1-e)] \\ &= xye + (1-e) = e + (1-e) \qquad \text{(by (1))} \\ &= 1. \end{aligned}$$

Thus v is a unit. Note that

$$\begin{aligned} fv = vf &= [xe + (1-e)][e + x(1-e)] \\ &= xe + x(1-e) = x. \end{aligned}$$

Then since $f^n = e$, it follows that $x^n = eu = ue$ where $u = v^n$ is a unit. □

In the case where $n = 1$ in the proof of Theorem 2.1 (that is, x is strongly regular), then $f = e$ and we have the following:

Proposition 2.2 *Let R be a ring. If x is a strongly regular element of R, then $x = eu = ue$ for some $e \in Id(R)$ and some $u \in U(R)$.*

We also note the following necessary condition for Euler rings.

Proposition 2.3 *If R is an Euler ring, then $U(R)$ is a torsion group.*

Proof. Let $u \in U(R)$. Since every element of R is a near idempotent, there exists a positive integer n such that u^n is an idempotent. Then $u^{2n} = u^n$ and hence,

$$u^n = u^{2n-n} = u^{2n}u^{-n} = u^n u^{-n} = 1.$$

Since u is arbitrary in $U(R)$, it follows that $U(R)$ is a torsion group. □

3 Euler rings

The main result in this section is as follows:

Theorem 3.1 *Let R be a ring. Then R is Euler if and only if R is strongly π-regular and $U(R)$ is a torsion group.*

Proof. Suppose that R is Euler. By Proposition 2.3, it follows readily that $U(R)$ is a torsion group. Now let $x \in R$ and let n be a positive integer such that x^n is an idempotent. Let $y = x^n$. Then $x^{2n}y = x^n$ and $xy = yx$. Hence R is strongly π-regular.

Conversely, suppose that R is strongly π-regular and $U(R)$ is a torsion group. Let $x \in R$. By Theorem 2.1, there exists a positive integer n such that

$$x^n = eu = ue$$

for some idempotent $e \in Id(R)$ and some unit $u \in U(R)$. Since $U(R)$ is a torsion group, there exists a positive integer m such that $u^m = 1$. Then $x^{nm} = e^m u^m = e$ is an idempotent of R. Since x is arbitrary in R, it follows that every element of R is a near idempotent. □

As a consequence of Theorem 3.1 we have the following:

Corollary 3.2 *A subring of an Euler ring is also Euler.*

Proof. Let R be an Euler ring and S a subring of R. For any $x \in S \leq R$, there exists a positive integer n such that $x^n \in Id(R)$. But $x^n \in S$ since S is a subring of R. Hence, $x^n \in Id(S)$ and it follows that S is also Euler. □

It is known that a subring of a strongly π-regular ring R is not necessarily strongly π-regular. However, if in addition $U(R)$ is torsion, then we have the following:

Corollary 3.3 *Let R be a strongly π-regular ring with $U(R)$ torsion. Then any subring of R is also strongly π-regular.*

Proof. Let S be a subring of R. Since R is Euler (by Theorem 3.1), it follows from Corollary 3.2 that S is also Euler. Hence, S is strongly π-regular by Theorem 3.1. □

Recall that a ring R is said to be *periodic* if for each $x \in R$ there are integers $m, n \geq 1$ such that $m \neq n$ and $x^m = x^n$. If R is an Euler ring it is easy to see that R is periodic. The converse is also true as has been shown in [2, Lemma 1]. In view of this and Theorem 3.1 we have the following corollary:

Corollary 3.4 *For a ring R the following conditions are equivalent:*

(a) R is Euler;

(b) R is periodic;

(c) R is strongly π-regular and $U(R)$ is a torsion group.

4 Exact-Euler rings

We obtain necessary and sufficient conditions for a ring to be exact-Euler as follows:

Theorem 4.1 *A ring R is exact-Euler if and only if R is strongly π-regular and Nil(R), $U(R)$ are of bounded index.*

Proof. Suppose first that R is exact-Euler. Then R is Euler and it follows readily from Theorem 3.1 that R is strongly π-regular. Let $u \in U(R)$ and $x \in$ Nil(R). Since R is exact-Euler, there is a fixed positive integer n such that u^n, $x^n \in Id(R)$. Then $u^n = u^{2n-n} = u^{2n}u^{-n} = u^n u^{-n} = 1$. Since u is arbitrary in $U(R)$, it follows that $U(R)$ is of bounded index. Let m be the smallest positive integer such that $x^m = 0$. Since $x^{kn} = x^n$ for any positive integer $k \geq 1$, then $m \leq n$. Hence, Nil(R) is of bounded index.

Conversely, suppose that R is strongly π-regular and Nil(R), $U(R)$ are of bounded index w, m, respectively. Let $x \in R$. Then there exist a positive integer n and an element $y \in R$ which commutes with x such that $x^n = x^{n+1}y$; thus $x^n = x^{2n}y^n$. Then since

$$\begin{aligned} x^{n+k} &= x^{2n+k}y^n = x^{n+k}(x^{n+1}y)y^n = x^{n+k+1}(x^{n+1}y)y^{n+1} = x^{2n+k+2}y^{n+2} \\ &= \cdots = x^{2(n+k)}y^{n+k} = x^{n+k+1}(x^{n+k-1}y^{n+k}) \end{aligned}$$

for any positive integer k, we may assume that $n > w$. Now since $(x^n y^n)^2 = x^{2n}y^{2n} = x^n y^n$, we have $x^n y^n \in Id(R)$ and hence, so is $1 - x^n y^n$. Note that $[x(1 - x^n y^n)]^n = x^n(1 - x^n y^n) = 0$. Thus, $[x(1 - x^n y^n)]^w = 0$ which gives us $x^w(1 - x^n y^n) = 0$. It follows that $x^w = x^{n+w}y^n = x^{2w}(x^{n-w}y^n)$; that is, x^w is strongly regular. By Proposition 2.2, $x^w = eu = ue$ for some $e \in Id(R)$ and some $u \in U(R)$. Thus, $x^{wm} = e^{wm}u^{wm} = e \in Id(R)$. Since x is arbitrary in R, this shows that R is exact-Euler. □

From the proof of Theorem 4.1 we have the following:

Proposition 4.2 *Suppose that R is a strongly π-regular ring and Nil(R), $U(R)$ are bounded above by w, $m \geq 1$ respectively. Then $x^{wm} \in Id(R)$ for each $x \in R$.*

As a consequence of Proposition 4.2 we have an algebraic proof of the following number-theoretic result:

Corollary 4.3 *Let $m = p_1^{\alpha_1} \dots p_n^{\alpha_n} \geq 2$ where the p_i are distinct primes and $\alpha_i \geq 1\,(i = 1, \dots, n)$. Let $k = \max\{\alpha_1, \dots, \alpha_n\}$ and let ϕ denote Euler's phi-function. Then $x^{k\phi(m)} \in Id(\mathbb{Z}_m)$ for each $x \in \mathbb{Z}_m$.*

Proof. It is well-known that $\mathbb{Z}_m$ is a strongly π-regular ring. Clearly, $\text{Nil}(\mathbb{Z}_m)$ is bounded above by k and $U(\mathbb{Z}_m)$ by $\phi(m)$. The result then follows by applying Proposition 4.2. □

5 Some related results

A ring R is said to be *unit regular* if for every $x \in R$, there exists a unit $u \in R$ such that $xux = x$. In [3], Ehrlich showed that if R is unit regular and 2 is a unit of R, then every element of R is a sum of two units of R. A ring R in which every element of R is a sum of two units of R is said to be an *$(s,2)$-ring* [5] (see also [4]). We say that R is an *$(s,2)$-π-ring* if for each element $x \in R$ there is an integer $n \geq 1$ such that x^n is a sum of two units of R. We also say that R is an *exact-$(s,2)$-π-ring* if there is a fixed integer $n \geq 1$ such that x^n is a sum of two units of R for every $x \in R$. Clearly, an exact-$(s,2)$-π-ring is $(s,2)$-π.

We obtain the following result:

Theorem 5.1 *(a) Let R be a strongly π-regular ring. Then R is an $(s,2)$-π-ring if and only if every element in $Id(R)$ is a sum of two units of R. In particular, if $2 \in U(R)$, then R is an $(s,2)$-π-ring.*

(b) Let R be an exact-Euler ring. Then R is an exact-$(s,2)$-π-ring if and only if every element in $Id(R)$ is a sum of two units of R. In particular, if $2 \in U(R)$, then R is an exact-$(s,2)$-π-ring.

Proof.

(a) Let $x \in R$. By Theorem 2.1, there is a positive integer n such that $x^n = eu = ue$

for some $e \in Id(R)$ and some $u \in U(R)$. Thus, R is an $(s,2)$-π-ring if each $e \in Id(R)$ is a sum of two units of R. The converse of this is clearly true. Now suppose that $2 \in U(R)$. Since $2e-1 \in U(R)$ for each $e \in Id(R)$ and $x^n = eu$, we have $x^n = 2^{-1}(1+(2e-1))u$ is a sum of two units of R.

(b) The necessity part of the first assertion is clearly true. For the converse, we only need to observe that there is a fixed positive integer n such that $x^n = e \in Id(R)$ for each $x \in R$. The final assertion in part (b) can be obtained by applying part (a) and the first assertion in this part. □

References

[1] G. Azumaya, Strongly π-regular rings, *J. Fac. Sci. Hokkaido Univ. Ser. I* **13** (1954), 34–39.

[2] H. E. Bell, A commutativity study for periodic rings, *Pac. J. Math.* **70** (1977), 29–36.

[3] G. Ehrlich, Unit regular rings, *Portugal. Math.* **27** (1968), 209–212.

[4] J. W. Fisher and R. L. Snider, Rings generated by their units, *J. Algebra* **42** (1976), 363–368.

[5] M. Henriksen, Two classes of rings that are generated by their units, *J. Algebra* **31** (1974), 182–193.